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# Jordan–Schwinger map, 3D harmonic oscillator constants of motion, and classical and quantum parameters characterizing electromagnetic wave polarization

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**Abstract**

In this work we introduce a generalization of the Jauch and Rohrlich quantum Stokes operators when the arrival direction from the source is unknown *a priori*. We define the generalized Stokes operators as the Jordan–Schwinger map of a triplet of harmonic oscillators with the Gell–Mann and Ne'eman matrices of the $SU(3)$ symmetry group. We show that the elements of the Jordan–Schwinger map are the constants of motion of the three-dimensional isotropic harmonic oscillator. Also, we show that the generalized Stokes operators together with the Gell–Mann and Ne'eman matrices may be used to expand the polarization matrix. By taking the expectation value of the Stokes operators in a three-mode coherent state of the electromagnetic field, we obtain the corresponding generalized classical Stokes parameters. Finally, by means of the constants of motion of the classical 3D isotropic harmonic oscillator we describe the geometrical properties of the polarization ellipse.

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## 1. Introduction

In both classical and quantum optics, Stokes parameters have proved to be intuitive and practical tools for characterizing the polarization state of light [1–6].

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A classical or quantum electromagnetic wave propagates, generally in an arbitrary but fixed direction in space. However, for the study of the polarization properties of the wave, knowledge of the propagation direction of the wave allows us to choose a coordinate system in such a way that the propagation is along the $z$-axis, and the polarization vector lies on the $x$–$y$-plane (i.e. it has only two polarization components) [7]. Also, knowledge of the propagation direction allows us to use 2D apparatus (polarizers, wave plate rotators, etc) placed perpendicular to the wave propagation direction to measure the polarization characteristics of the wave. The works [8–10] were done under the assumption that the arrival direction from the source of the electromagnetic wave was known. In [8] Stokes studied the polarization properties of a quasi-monochromatic plane wave of light in an arbitrary polarization state by introducing four quantities, known since then as the Stokes parameters. Wiener used the $2 \times 2$ identity matrix together with the Pauli matrices as a basis to expand the coherence tensor [9]. Fano [10] showed that the Stokes parameters are the expansion coefficients of the coherence tensor. Stokes parameters obtained under an *a priori* knowledge of the propagation direction will be referred to in this work as the usual classical or quantum Stokes parameters, which are well described in [7, 11] and [12], respectively.

When we do not know *a priori* the propagation direction of the wave, we no longer have an adequate choice of coordinate system as above, and thus, in general, the three components of the polarization vector are non-zero. In this case, the three-dimensional coherence tensor must be used to obtain a complete polarization characterization [13, 14]. Roman [13] used the basis of nine Hermitian $3 \times 3$ matrices which constitute a Kemmer algebra to define the generalized Stokes parameters as the expansion coefficients of the correlation matrix. In [14], Carozzi *et al* defined the generalized Stokes parameters as the expansion coefficients of the spectral density tensor in terms of the $SU(3)$ Gell–Mann and Ne'eman matrices.

In this work we introduce a generalization of the Jauch and Rohrlich quantum Stokes operators when the arrival direction from the source is unknown *a priori*. For simplicity, we study the case of a monochromatic quantized plane electromagnetic wave that propagates in a fixed but arbitrary direction in space. Also, we will set $\hbar = \omega = \mu = 1$, where $\mu$ is the mass of each 1D harmonic oscillator and $\omega$ is the angular frequency of either the electromagnetic wave or each harmonic oscillator. In section 2, we define the generalized quantum Stokes operators as the Jordan–Schwinger map of a triplet of harmonic oscillators with the Gell–Mann and Ne'eman $\lambda_i$ matrices of the $SU(3)$ symmetry group. We show that the elements of the Jordan–Schwinger map are the constants of motion of the quantum 3D isotropic harmonic oscillator. Also, we show that the generalized Stokes operators together with the $\lambda_i$ matrices may be used to expand the polarization matrix. In section 3, we take the expectation value of the generalized quantum Stokes operators in a three-mode coherent state of the electromagnetic field to obtain the corresponding generalized classical Stokes parameters. In section 4, by means of the classical constants of motion of the 3D isotropic harmonic oscillator we describe the geometrical properties of the polarization ellipse. Finally, in section 5, we give some concluding remarks.

## 2. Jordan–Schwinger map and the harmonic oscillator constants of motion

Usual classical Stokes parameters are defined as the expansion coefficients of the polarization matrix [11, 15] as

$$J_{2D} = \frac{1}{2}\sum_{i=0}^{3} \sigma_i s_i \tag{1}$$

where $s_i$ are the four Stokes parameters, $\sigma_0 = \mathbf{1}_{2\times 2}$ and $\sigma_i$, $i = 1, 2, 3$, are the Pauli matrices. Since the $\sigma_i$ matrices satisfy that $\mathrm{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$ and $\mathrm{Tr}(\sigma_0 \sigma_j) = 0$, then

$$\mathrm{Tr}(J_{2\mathrm{D}}\sigma_j) = s_j. \tag{2}$$

### *2.1. Usual quantum Stokes operators*

The usual Stokes operators for a quantized plane electromagnetic wave that propagates along the $z$-axis are defined as [12]

$$\begin{aligned} S_0 &= a^\dagger \sigma_0 a = a_1^\dagger a_1 + a_2^\dagger a_2 & \qquad S_1 &= a^\dagger \sigma_1 a = a_1^\dagger a_2 + a_2^\dagger a_1 \\ S_2 &= a^\dagger \sigma_2 a = \mathrm{i}\left(-a_1^\dagger a_2 + a_2^\dagger a_1\right) & \qquad S_3 &= a^\dagger \sigma_3 a = a_1^\dagger a_1 - a_2^\dagger a_2 \end{aligned} \tag{3}$$

where $a_j^\dagger$ and $a_j$, $j = 1, 2$, are the creation and annihilation operators of the $j$th harmonic oscillator defined as

$$a_j^\dagger = \frac{1}{\sqrt{2}}(x_j - \mathrm{i}p_j) \qquad a_j = \frac{1}{\sqrt{2}}(x_j + \mathrm{i}p_j) \tag{4}$$

with $\left[a_1, a_1^\dagger\right] = \left[a_2, a_2^\dagger\right] = 1$ and

$$a^\dagger = \left(a_1^\dagger, a_2^\dagger\right) \qquad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{5}$$

We note that equations (3) are a particular case of the Jordan–Schwinger map with two kinematically independent bosons [16].

In the rest of this paper, the following observation is of fundamental importance: the quantities in (3) coincide with the constants of motion of the 2D isotropic harmonic oscillator with Hamiltonian $H_{2\mathrm{D}} = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$. In fact, we can show that

$$[S_i, H_{2\mathrm{D}}] = 0 \qquad i = 0, 1, 2, 3. \tag{6}$$

The commutation relations of the Stokes operators are immediately obtained from the properties of the Jordan–Schwinger map [16]. This leads us to the $SU(2)$ Lie algebra

$$\left[\frac{S_\ell}{2}, \frac{S_m}{2}\right] = \mathrm{i}\epsilon_{\ell mn}\frac{S_n}{2} \qquad \ell, m, n = 1, 2, 3 \tag{7}$$

where $\epsilon_{\ell mn}$ is the totally antisymmetric tensor.

We note that the angular momentum and the energy minus the zero-point energy of the 2D isotropic harmonic oscillator are equal to

$$L_z = S_2 \qquad H_{2\mathrm{D}} - 1 = S_0 \tag{8}$$

respectively. According to Jauch and Rorhlich [12], the spin of the photon is given by $S_2$ and it is along the direction of propagation. Therefore, the first equality in (8) means that the angular momentum of the 2D isotropic harmonic oscillator is equal to the spin operator of the photon.

Using equations (3), we can write the polarization matrix in terms of the constants of motion of the 2D isotropic harmonic oscillator (usual quantum Stokes operators) as

$$J_{2\mathrm{D}} = \frac{1}{2}\begin{pmatrix} \langle S_0\rangle_\alpha + \langle S_3\rangle_\alpha & \langle S_1\rangle_\alpha + \mathrm{i}\langle S_2\rangle_\alpha \\ \langle S_1\rangle_\alpha - \mathrm{i}\langle S_2\rangle_\alpha & \langle S_0\rangle_\alpha - \langle S_3\rangle_\alpha \end{pmatrix} \tag{9}$$

where the $\langle S_i\rangle_\alpha$, $i = 0, 1, 2, 3, 4$, are the classical limits of the Stokes operators, and they will be found in section 3 by taking their expectation values in coherent or semiclassical states of the electromagnetic field.

The physical and geometrical implications of the equality between the Stokes operators and the constants of motion of the 2D isotropic harmonic oscillator are extensively discussed in [17].

### 2.2. Generalized Stokes operators

When the direction of arrival from the source is unknown *a priori*, we generalize the quantum Stokes operators as follows. By using the Gell–Mann and Ne'eman $\lambda_i$ matrices of the $SU(3)$ symmetry group [18] and the triplet of independent harmonic oscillators (three independent bosons) $a^\dagger = \left(a_1^\dagger, a_2^\dagger, a_3^\dagger\right)$, we define the generalized quantum Stokes operators as the Jordan–Schwinger map $\Sigma_i = a^\dagger \lambda_i a$. These are explicitly given by

$$\begin{aligned}
&\Sigma_0 = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \qquad &&\Sigma_1 = a_1^\dagger a_2 + a_2^\dagger a_1 \qquad &&\Sigma_2 = \mathrm{i}\left(-a_1^\dagger a_2 + a_2^\dagger a_1\right)\\
&\Sigma_3 = a_1^\dagger a_1 - a_2^\dagger a_2 &&\Sigma_4 = a_1^\dagger a_3 + a_3^\dagger a_1 &&\Sigma_5 = \mathrm{i}\left(a_3^\dagger a_1 - a_1^\dagger a_3\right)\\
&\Sigma_6 = a_2^\dagger a_3 + a_3^\dagger a_2 &&\Sigma_7 = \mathrm{i}\left(a_3^\dagger a_2 - a_2^\dagger a_3\right) &&\Sigma_8 = \tfrac{1}{\sqrt{3}}\left(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3\right)
\end{aligned} \tag{10}$$

where we have used $\lambda_0 = \mathbf{1}_{3\times 3}$.

From the commutation relations $\left[a_1, a_1^\dagger\right] = \left[a_2, a_2^\dagger\right] = \left[a_3, a_3^\dagger\right] = 1$, we show that the generalized quantum Stokes operators are the constants of motion of the 3D isotropic harmonic oscillator with Hamiltonian $H_{3\mathrm{D}} = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 + \frac{3}{2}$, i.e.

$$[\Sigma_i, H_{3\mathrm{D}}] = 0 \qquad i = 0, \ldots, 8. \tag{11}$$

Also, by the properties of the Jordan–Schwinger map [16], we show that the generalized quantum Stokes operators satisfy the commutation rules of the $SU(3)$ Lie algebra

$$\left[\frac{\Sigma_\ell}{2}, \frac{\Sigma_m}{2}\right] = \mathrm{i} f_{\ell m n}\frac{\Sigma_n}{2} \qquad \ell, m, n = 1, \ldots, 8 \tag{12}$$

where the structure constants $f_{\ell mn}$ are totally antisymmetric under exchange of any two indices and are given by

$$\begin{aligned}
&f_{123} = 1 \qquad &&f_{147} = \tfrac{1}{2} \qquad &&f_{156} = -\tfrac{1}{2}\\
&f_{246} = \tfrac{1}{2} &&f_{257} = \tfrac{1}{2} &&f_{345} = \tfrac{1}{2}\\
&f_{367} = -\tfrac{1}{2} &&f_{458} = \tfrac{\sqrt{3}}{2} &&f_{678} = \tfrac{\sqrt{3}}{2}.
\end{aligned} \tag{13}$$

A careful analysis leads us to show that the angular momentum operator $\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}}$ and the energy operator of the 3D isotropic harmonic oscillator are contained in the generalized quantum Stokes operators. Explicitly, we can show that

$$L_1 = \Sigma_7 \qquad L_2 = -\Sigma_5 \qquad L_3 = \Sigma_2 \qquad H_{3\mathrm{D}} - \tfrac{3}{2} = \Sigma_0. \tag{14}$$

Because of the first three equalities in (14), the generalization of the remarks after equations (8) means that the angular momentum of the 3D isotropic harmonic oscillator essentially is equal to the spin operator of the photon.

We generalize the definition of the polarization matrix as follows,

$$J_{3\mathrm{D}} = \frac{1}{3}\lambda_0\langle\Sigma_0\rangle_\alpha + \frac{1}{2}\sum_{i=1}^{8}\lambda_i\langle\Sigma_i\rangle_\alpha \tag{15}$$

where, again, $\langle\Sigma_i\rangle_\alpha$, $i = 0, \ldots, 8$, are the classical limits of the generalized quantum Stokes operators. In the following section these are shown to be the expectation values of the operators $\Sigma_i$ in a coherent state of the electromagnetic field.

Since the $\lambda_i$ matrices are such that $\mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}$ and $\mathrm{Tr}(\lambda_0\lambda_i) = 0, i, j = 1, \ldots, 8,$ then

$$\mathrm{Tr}(J_{3\mathrm{D}}\lambda_j) = \frac{1}{2}\,\mathrm{Tr}\left(\sum_{i=1}^{8}\lambda_i\lambda_j\langle\Sigma_i\rangle_\alpha\right) = \langle\Sigma_j\rangle_\alpha. \tag{16}$$

By using equations (10) and (15), the polarization matrix in terms of the 3D isotropic harmonic oscillator constants of motion (or the generalized quantum Stokes operators) takes the form

$$J_{3\mathrm{D}} = \begin{pmatrix} \frac{1}{3}\langle\Sigma_0\rangle_\alpha + \frac{1}{2}\langle\Sigma_3\rangle_\alpha + \frac{1}{2\sqrt{3}}\langle\Sigma_8\rangle_\alpha & \frac{1}{2}\langle\Sigma_1\rangle_\alpha - \mathrm{i}\frac{1}{2}\langle\Sigma_2\rangle_\alpha & \frac{1}{2}\langle\Sigma_4\rangle_\alpha - \mathrm{i}\frac{1}{2}\langle\Sigma_5\rangle_\alpha \\ \frac{1}{2}\langle\Sigma_1\rangle_\alpha + \mathrm{i}\frac{1}{2}\langle\Sigma_2\rangle_\alpha & \frac{1}{3}\langle\Sigma_0\rangle_\alpha - \frac{1}{2}\langle\Sigma_3\rangle_\alpha + \frac{1}{2\sqrt{3}}\langle\Sigma_8\rangle_\alpha & \frac{1}{2}\langle\Sigma_6\rangle_\alpha - \mathrm{i}\frac{1}{2}\langle\Sigma_7\rangle_\alpha \\ \frac{1}{2}\langle\Sigma_4\rangle_\alpha + \mathrm{i}\frac{1}{2}\langle\Sigma_5\rangle_\alpha & \frac{1}{2}\langle\Sigma_6\rangle_\alpha + \mathrm{i}\frac{1}{2}\langle\Sigma_7\rangle_\alpha & \frac{1}{3}\langle\Sigma_0\rangle_\alpha - \frac{1}{\sqrt{3}}\langle\Sigma_8\rangle_\alpha \end{pmatrix}. \tag{17}$$

We observe that the $\lambda_0$ coefficient in equation (15) is such that equation (17) reduces to $J_{2\mathrm{D}}$ when the propagation direction of the plane electromagnetic wave is chosen to be along the $z$-axis. Also, we note that our definition of $J_{3\mathrm{D}}$ is such that the trace of $J_{2\mathrm{D}}$ and $J_{3\mathrm{D}}$ remains invariant.

It is important to note that the polarization matrix (15) can be defined formally for purely quantum states. This means that it can be defined without taking the expectation values in a semiclassical state of the electromagnetic field of the Stokes operators $\Sigma_i$. In this way, equation (16) becomes $\mathrm{Tr}(J_{3\mathrm{D}}\lambda_j) = \Sigma_j$. However, the implications of this definition are beyond the scope of this work.

## 3. Generalized classical Stokes parameters

We will obtain the classical limit for the generalized quantum Stokes operators. To do this, we proceed as in [19] to obtain the classical limit of the usual Stokes operators by taking the expectation value of the operators (3) in a two-mode coherent state of the electromagnetic field. In our case, we compute the mean value of the generalized quantum Stokes operators (10) in the three-mode coherent state of the electromagnetic field

$$|\alpha_1, \alpha_2, \alpha_3\rangle = \sum_{n_1,n_2,n_3=0}^{\infty}\frac{\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}}{\sqrt{n_1!n_2!n_3!}}|n_1, n_2, n_3\rangle. \tag{18}$$

This leads us to the generalized classical Stokes parameters

$$\begin{aligned} \langle\Sigma_0\rangle_\alpha &= |\alpha_{01}|^2 + |\alpha_{02}|^2 + |\alpha_{03}|^2 & \langle\Sigma_1\rangle_\alpha &= 2|\alpha_{01}||\alpha_{02}|\cos\Delta_{21} \\ \langle\Sigma_2\rangle_\alpha &= 2|\alpha_{01}||\alpha_{02}|\sin\Delta_{21} & \langle\Sigma_3\rangle_\alpha &= |\alpha_{01}|^2 - |\alpha_{02}|^2 \\ \langle\Sigma_4\rangle_\alpha &= 2|\alpha_{01}||\alpha_{03}|\cos\Delta_{31} & \langle\Sigma_5\rangle_\alpha &= 2|\alpha_{01}||\alpha_{03}|\sin\Delta_{31} \\ \langle\Sigma_6\rangle_\alpha &= 2|\alpha_{02}||\alpha_{03}|\cos\Delta_{32} & \langle\Sigma_7\rangle_\alpha &= 2|\alpha_{02}||\alpha_{03}|\sin\Delta_{32} \\ \langle\Sigma_8\rangle_\alpha &= |\alpha_{01}|^2 + |\alpha_{02}|^2 - 2|\alpha_{03}|^2 \end{aligned} \tag{19}$$

where $\alpha_i = |\alpha_{0i}|\exp(\mathrm{i}\phi_i)$, and $\Delta_{ij} \equiv \phi_i - \phi_j$ is the classical phase difference.

It will be shown in section 4 that equation (19) represents the Stokes parameters for three classical oscillations of amplitudes $|\alpha_{0i}|$ and phases $\phi_i$. It is immediate to note that these equalities reduce to the usual classical Stokes parameters, when the amplitude and phase of the third oscillation vanish.

## 4. Classical Stokes parameters, classical 3D isotropic harmonic oscillator constants of motion and the geometrical properties of the polarization ellipse

The classical 3D isotropic harmonic oscillator is a particle that moves under the force

$$\mathbf{F} = -\mathbf{r}. \tag{20}$$

By solving Newton's second law and imposing the initial conditions $\mathbf{r}_{t=0} = \mathbf{x_o}$ and $\mathbf{v}_{t=0} = \mathbf{v_o}$, we obtain the solutions

$$x_i = a_i \cos t + b_i \sin t \qquad i = 1, 2, 3 \tag{21}$$

where $a_i = x_{oi}$ and $b_i = v_{oi}$. It is easy to see that these solutions satisfy the ellipsoid equation

$$\begin{aligned}x_1^2\left(a_2^2 + b_2^2 + a_3^2 + b_3^2\right) &+ x_2^2\left(a_1^2 + b_1^2 + a_3^2 + b_3^2\right) + x_3^2\left(a_2^2 + b_2^2 + a_1^2 + b_1^2\right) \\ &- 2x_1x_2(a_1a_2 + b_1b_2) - 2x_2x_3(a_2a_3 + b_2b_3) - 2x_1x_3(a_1a_3 + b_1b_3) \\ &= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2.\end{aligned} \tag{22}$$

This means that the orbit of the classical 3D isotropic harmonic oscillator is contained in the ellipsoid. Moreover, since the classical 3D isotropic harmonic oscillator potential has spherical symmetry, its orbit is restricted to be on the orthogonal plane to the classical angular momentum $\mathbf{L}_{cl} = \mathbf{r} \times \mathbf{p}$. Thus, the elliptic orbit of the classical 3D isotropic harmonic oscillator is the curve given by the intersection of the ellipsoid (22) and the plane orthogonal to $\mathbf{L}_{cl}$, which contains the origin of coordinates.

Equation (21) can be written in an oscillation form as

$$x_i = |\alpha_{0i}| \sin(t + \phi_i) \tag{23}$$

with

$$a_i = |\alpha_{0i}| \sin\phi_i \qquad b_i = |\alpha_{0i}| \cos\phi_i. \tag{24}$$

These equalities imply that

$$\alpha_{0i}^2 = a_i^2 + b_i^2 \qquad \sin\phi_i = \frac{a_i}{\sqrt{a_i^2 + b_i^2}} \qquad \cos\phi_i = \frac{b_i}{\sqrt{a_i^2 + b_i^2}}. \tag{25}$$

The amplitudes and phases of the three classical oscillations of equation (23) depend on the initial conditions $a_i$ and $b_i$ according to equations (25). Thus, if we substitute equations (25) into equations (19), we incorporate the initial conditions in the generalized classical Stokes parameters (constants of motion of the classical 3D isotropic harmonic oscillator). In particular, at $t = 0$, the constant of motion of the angular momentum vector is

$$\mathbf{L}_{cl} = (\langle\Sigma_7\rangle_\alpha, -\langle\Sigma_5\rangle_\alpha, \langle\Sigma_2\rangle_\alpha) = \mathbf{a} \times \mathbf{b} \tag{26}$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and the ellipsoid equation (22) turns out to be

$$\begin{aligned}x_1^2\left(\frac{4\langle\Sigma_0\rangle_\alpha - \langle\Sigma_8\rangle_\alpha}{6} - \frac{1}{2}\langle\Sigma_3\rangle_\alpha\right) &+ x_2^2\left(\frac{4\langle\Sigma_0\rangle_\alpha - \langle\Sigma_8\rangle_\alpha}{6} + \frac{1}{2}\langle\Sigma_3\rangle_\alpha\right) \\ &+ x_3^2\left(\frac{2\langle\Sigma_0\rangle_\alpha + \langle\Sigma_8\rangle_\alpha}{3}\right) - x_1x_2\langle\Sigma_1\rangle_\alpha - x_2x_3\langle\Sigma_6\rangle_\alpha - x_1x_3\langle\Sigma_4\rangle_\alpha \\ &= \frac{1}{4}\left(\langle\Sigma_7\rangle_\alpha^2 + \langle\Sigma_5\rangle_\alpha^2 + \langle\Sigma_2\rangle_\alpha^2\right).\end{aligned} \tag{27}$$

Following the definition of the Euler angles in [20], we perform a rotation such that the direction of the new $x_1$-axis coincides with that of the line of nodes, and the direction of the new $x_3$-axis coincides with that of $\mathbf{L}_{cl}$. The direction of the line of nodes (direction of the intersection line between the orbit and the $x_1$–$x_2$-plane) is found by a unitary vector in the $x_1$–$x_2$-plane, perpendicular to $\mathbf{L}_{cl} = (\langle\Sigma_7\rangle_\alpha, -\langle\Sigma_5\rangle_\alpha, \langle\Sigma_2\rangle_\alpha)$. This leads us to

$$\begin{aligned}\sin\phi = n_x &= \pm\frac{\langle\Sigma_7\rangle_\alpha}{\sqrt{\langle\Sigma_7\rangle_\alpha^2 + \langle\Sigma_5\rangle_\alpha^2}}\\ \cos\phi = n_y &= \mp\frac{\langle\Sigma_5\rangle_\alpha}{\sqrt{\langle\Sigma_7\rangle_\alpha^2 + \langle\Sigma_5\rangle_\alpha^2}}.\end{aligned} \tag{28}$$

The orthogonality between $\mathbf{L}_{cl}$ and the ellipse plane leads to

$$\cos\theta = \frac{\langle\Sigma_2\rangle_\alpha}{\sqrt{\left(\langle\Sigma_7\rangle_\alpha^2 + \langle\Sigma_5\rangle_\alpha^2 + \langle\Sigma_2\rangle_\alpha^2\right)}}. \tag{29}$$

On the other hand, it is well known that the constants of motion of the classical 3D isotropic harmonic oscillator, in addition to the energy and the angular momentum are given by the symmetric Runge–type tensor [21]

$$A_{ij} = \tfrac{1}{2}(p_i p_j + \omega^2 x_i x_j) \qquad i, j = 1, 2, 3. \tag{30}$$

It can be shown that the contraction of this equation with the components of $\mathbf{L}_{cl}$ yields zero. This means that all the geometrical characteristics of the orbit must be determined by $A_{ij}$. In fact, we can show that

$$\begin{aligned}2A_{11} &= \frac{2\langle\Sigma_0\rangle_\alpha + \langle\Sigma_8\rangle_\alpha}{6} + \frac{\langle\Sigma_3\rangle_\alpha}{2} & 2A_{22} &= \frac{2\langle\Sigma_0\rangle_\alpha + \langle\Sigma_8\rangle_\alpha}{6} - \frac{\langle\Sigma_3\rangle_\alpha}{2}\\ 2A_{33} &= \frac{\langle\Sigma_0\rangle_\alpha - \langle\Sigma_8\rangle_\alpha}{3} & 2A_{12} &= \frac{1}{2}\langle\Sigma_1\rangle_\alpha\\ 2A_{13} &= \frac{1}{2}\langle\Sigma_4\rangle_\alpha & 2A_{23} &= \frac{1}{2}\langle\Sigma_6\rangle_\alpha.\end{aligned} \tag{31}$$

This shows that the geometrical properties of the polarization ellipse exclusively depend on the generalized classical Stokes parameters. It can be shown that the eigenvalues of $A_{ij}$ depend only on the energy and the magnitude of the angular momentum of the classical 3D isotropic harmonic oscillator [21]. The tensor $A_{ij}$ has an eigenvector in the direction of the angular momentum, and its other two eigenvectors are in the directions of the principal axis of the elliptical orbit [21]. Also, the eigenvectors of a symmetric rank-two tensor are determined by its eigenvalues as well as its components [22]. The above remarks lead us to conclude that the principal axis directions of the elliptical orbit are completely determined by the generalized classical Stokes parameters. Also, since $\mathbf{L}_{cl}$ is orthogonal to the polarization ellipse, then $\mathbf{L}_{cl}$ points along the propagation direction of the electromagnetic wave.

## 5. Concluding remarks

This work links quantum optics to classical optics by means of quantum mechanics and it is a useful extension of the generalized classical Stokes parameters into the quantum domain.

Although there are already treatments of the classical Stokes parameters in the case of an *a priori* unknown direction of the electromagnetic wave propagation [13, 14], our treatment results to be novel in the following aspects. We have introduced a generalization of the quantum Stokes parameters of Jauch *et al* [12] using the Jordan–Schwinger map, three independent bosons and the Gell–Mann and Ne'eman $SU(3)$ symmetry group matrices. It was shown that the generalized quantum Stokes operators turn out to be the expansion coefficients of the polarization matrix in terms of the Gell–Mann and Ne'eman $SU(3)$ matrices.

The semiclassical limit of the generalized Stokes operators was achieved by taking their expectation values in a three-mode coherent state of the electromagnetic field. Thus, our treatment in the quantum domain is more general than those given in [13, 14], which are restricted to the classical aspects of electromagnetic polarization.

We described by means of the classical 3D isotropic harmonic oscillator constants of motion, the geometrical properties of the polarization ellipse. Particularly, we showed that the ellipsoid coefficients and the symmetric Runge-type tensor of the classical 3D isotropic harmonic oscillator are completely determined by the generalized classical Stokes parameters. Also, we showed that the first two Euler angles are intimately related to the components of the orbital angular momentum of the classical 3D isotropic harmonic oscillator.

Finally, we emphasize that *our generalization* provides six independent generalized classical Stokes parameters. This is because going from (22) to (27), all of them were written in terms of the six parameters, $a_i$ and $b_i$, $i = 1, 2, 3$ which contain the initial conditions of the classical 3D isotropic harmonic oscillator.

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## References

[1] Shurcliff W A 1966 *Polarized Light: Production and Use* (Cambridge, MA: Harvard University Press)
[2] Ramachandran G N and Ramaseshan S 1961 *Encyclopedia of Physics* ed S Flügge (Berlin: Springer)
[3] Agarwal G S and Chaturvedi S 2003 *J. Mod. Opt.* **50** 711
[4] Lehner J, Leonhardt U and Paul H 1996 *Phys. Rev.* A **53** 2727
[5] Abouraddy A F, Sergienko A V, Saleh B E A and Teich M C 2002 *Opt. Commun.* **201** 93
[6] Jaeger G, Teodorescu-Frumosu M, Sergienko A V, Saleh B E A and Teich M C 2003 *Phys. Rev.* A **67** 032307
[7] Jackson J D 1975 *Classical Electrodynamics* (New York: Wiley) p 273
[8] Stokes G G 1852 *Trans Cambridge Philos. Soc.* **9** 399 (Reprinted in Stokes G G 1966 *Mathematical and Physical Papers* New York, London: Johnson Reprint Corporation)
[9] Wiener N 1930 *Acta Math.* **55** 117 Reprinted in Wiener N 1964 *Generalized Harmonic Analysis and Tauberian Theorems* Cambridge, MA: MIT Press)
[10] Fano U 1954 *Phys. Rev.* **93** 121
[11] Simmons J W and Guttmann M J 1970 *States, Waves and Photons: A Modern Introduction to Light* (New York: Wiley) p 74
[12] Jauch J M and Rohrlich F 1976 *The Theory of Photons and Electrons* (Berlin: Springer) p 41
[13] Roman P 1959 *Nuovo Cimento* **13** 974
[14] Carozzi T, Karlsson R and Bergman J 2000 *Phys. Rev.* E **65** 2024
[15] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge, MA: Cambridge University Press) p 349
[16] Biedenharn L C and Louck J D 1981 *Angular Momentum in Quantum Physics* (Reading, MA: Addison-Wesley) p 213
[17] Mota R D, Xicoténcatl M A and Granados V D 2003 *Reporte interno No. 340* (Depto. de Matemáticas, CINVESTAV) submitted
[18] Gell–Mann M and Ne'eman Y 1964 *The Eight-fold Way* (New York: Benjamin)
[19] Tanas R and Gantsog T S 1992 *Opt. Commun.* **87** 369
[20] Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison-Wesley) p 143
[21] Fradkin D M 1967 *Prog. Theor. Phys.* **37** 798
Fradkin D M 1965 *Am. J. Phys.* **33** 207
[22] Birisenko A I and Tarapov I E 1979 *Vector and Tensor Analysis* (New York: Dover) p 115